# Investigation: Product of Polynomials – Possible Responses

**Component 1**

In this component you will consider graphs of cubic polynomials.

1. Consider the family of curves of the form $y=(x-a)(x-b)(x-c)$ where $a,b,c$ are real numbers $(a,b,c\in R)$.
2. i. By selecting your own values for $a, b, c,$ where $a\ne b\ne c$, sketch 3 cubic graphs of the above form. Label your axial intercepts with coordinates. Also write the equation for each corresponding graph.

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| --- | --- | --- |
| $$a=1, b=2, c=3$$$$y=(x-1)(x-2)(x-3)$$ | $$a=-1, b=2, c=3$$$$y=(x+1)(x-2)(x-3)$$ | $$a=-2, b=1, c=2$$$$y=(x+2)(x-1)(x-2)$$ |

1. Comment on any similarities/differences between your graphs.

Similarities:

* They all have 3 $x$-intercepts
* They all have 2 turning points
* They all have the same shape

Differences:

* Some have positive $y$-intercepts while others have negative $y$-intercepts
* Some have all positive $x$-intercepts, some have all negative $x$-intercepts, and some have both positive and negative $x$-intercepts
1. Discuss how $a,b,c$ affect the key features of the graph.

The values pf $a, b, c$ correspond to the values of the $x$-intercepts

1. What happens to the shape and key features of the cubic graph if $a=b=c$? Investigate. Provide 3 examples to support your ideas/conjectures. Label your axial intercepts with coordinates. Also write the equation for each corresponding graph.
Comment on any similarities/differences. Try to generalise your observations.

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| --- | --- | --- |
| $$a=b=c=1$$$$y=\left(x-1\right)^{3}$$ | $$a=b=c=-2$$$$y=\left(x+2\right)^{3}$$ | $$a=b=c=-\frac{1}{2}$$$$y=\left(x+\frac{1}{2}\right)^{3}$$ |

Similarities:

* All have the same shape
* All have just one $x$-intercept
* All have a stationary point of inflection at the $x$-intercept

Differences:

* Some have positive $y$-intercepts while others have negative $y$-intercepts
* Some have positive $x$-intercepts, some have negative $x$-intercepts

Generalisations:

* All graphs of the form $y=\left(x-a\right)^{3}$ will have a stationary point of inflection at the $x$-intercept
* The graph/stationary point of inflection/$x$-intercept will move to the right if $a>0$ and will move to the left if $a<0$
* The $y$-interept of $y=\left(x-a\right)^{3}$ will be positive if $a<0$ and negative if $a>0$
1. What happens to the shape and key features of the cubic graph if $a=b\ne c$? Investigate. Provide 3 examples to support your ideas/conjectures. Label your axial intercepts with coordinates. Also write the equation for each corresponding graph.
Comment on any similarities/differences. Try to generalise your observations.

|  |  |  |
| --- | --- | --- |
| $$a=b=1, c=2$$$$y=\left(x-1\right)^{2}(x-2)$$ | $$a=b=-1, c=3$$$$y=\left(x+1\right)^{2}(x-3)$$ | $$a=b=-1, c=-2$$$$y=\left(x+1\right)^{2}\left(x+2\right)$$ |

Similarities:

* All have the same shape
* All have just two $x$-intercept
* Some students may use CAS technology to identify that all of these graphs will have a (non-stationary) point of inflection

Differences:

* Some have positive $y$-intercepts while others have negative $y$-intercepts
* Some have positive $x$-intercepts, some have negative $x$-intercepts

Generalisations:

* All graphs of the form $y=\left(x-a\right)^{2}(x-c)$ will have a turning point at $x=a$ (or at the point $\left(a,0\right)$), and the graph will cut through the $x$-axis at $x=c$ (or the point $\left(c,0\right)$ ).
* The graph will have a positive $y$-intercept if $c$ is negative, and a negative $y$-intercept if $c$ is positive
1. Consider the family of curves of the form $y=(x+1)(ax^{2}+bx+c)$,
where $a,b,c$ are non-zero real numbers.
Investigate how $a,b,c$ affect the shape and key features of the graph.
Provide examples to support your ideas/conjectures. Summarise/describe your observations, and try to generalise your observations.
* $c$ corresponds to the $y$-intercepts
* All graphs will pass through the $x$-intercept $\left(1,0\right)$
* $Δ=b^{2}-4ac$ will determine the number of intercepts of the graph
	+ $Δ<0$ will give one $x$-intercept at $\left(1,0\right)$
	+ $Δ=0$ will give two $x$-intercepts
	+ $Δ>0$ will give three $x$-intercepts

**Component 2**

In this component you will consider graphs of the form $y=x^{m}\left(h-x\right)^{n}$, where $h$ is non-zero real number and $m,n\in \left\{0, 1, 2, 3, 4\right\}$

1. Consider the case where $h=2$.
2. Sketch the following graphs, labelling all key features.

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| $$m=0, n=0$$ | $$y=x^{0}\left(2-x\right)^{0}$$ | $$m=1, n=1$$ | $$y=x(2-x)$$ |
|  |  |
| $$m=2, n=1$$ | $$y=x^{2}\left(2-x\right)$$ | $$m=1, n=2$$ | $$y=x\left(2-x\right)^{2}$$ |
|  |  |
| $$m=3, n=1$$ | $$y=x^{3}\left(2-x\right)$$ | $$m=1, n=3$$ | $$y=x\left(2-x\right)^{3}$$ |
|  |  |
| $$m=2, n=2$$ | $$y=x^{2}\left(2-x\right)^{2}$$ |  |
|  |  |

1. Comment on any similarities/differences between your graphs.

Similarities:

* All graphs (except for the case of $m=n=0$) pass through the points $\left(0,0\right)$ and $\left(2,0\right)$/have two $x$-intercepts
* The graphs of $m=2 \& n=1$ and $m=1$ & $n=2$ (i.e. $m+n=3$) have a cubic shape.
* The graphs of $m=3 \& n=1$ and $m=1$ & $n=3$ have a point of inflection at one of the $x$-intercepts (at the intercept whose factor has degree 3).
* There is a turning point when at least one of the powers is greater than or equal to 2 (2 or more).
* There is a stationary point of inflection when one of the powers is 3

Differences:

* ~~Some have positive~~ $y$~~-intercepts while others have negative~~ $y$~~-intercepts~~
* ~~Some have positive~~ $x$~~-intercepts, some have negative~~ $x$~~-intercepts~~
1. Discuss how $m$ and $n$ affect:
* The behaviour/shape of the graph. Provide examples to support your ideas/conjectures (select a different value of $h$).
* The number and nature of any turning points/point of inflection. Provide examples to support your ideas/conjectures (select a different value of $h$).
* $m \& n$ affect the behaviour of the graph at the $x$-intercepts
* When $m$ or $n$ is odd, the graph cuts through the $x$-intercept
	+ If $m$ or $n$ is 3, there is also a stationary point of inflection at the intercept that has the power of 3
* When $m$ or $n$ is even, the graph will have a turning point at the $x$-interept that corresponds to that power (factor).
1. What would happen to the graph and key features if $h$ was negative? Investigate.
Students could try a particular value of $h$ (or values of $h$), compare with their earlier results and then notice any patterns emerging to generalise their results/observations.

Consider $h=-2$

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| --- | --- | --- | --- |
| $$m=1, n=1$$ | $$y=x(-2-x)$$$$y=-x(x+2)$$ | $$m=1, n=2$$ | $$y=x\left(-2-x\right)^{2}$$$$y=x\left(x+2\right)^{2}$$ |
|  |  |
| $$m=2, n=1$$ | $$y=x^{2}\left(-2-x\right)$$$$y=-x^{2}\left(x+2\right)$$ | $$m=1, n=3$$ | $$y=x\left(-2-x\right)^{3}$$$$y=-x\left(x+2\right)^{3}$$ |
|  |  |
| $$m=3, n=1$$ | $$y=x^{3}\left(-2-x\right)$$$$y=-x^{3}\left(x+2\right)$$ | $$m=2, n=2$$ | $$y=x^{2}\left(-2-x\right)^{2}$$$$y=x^{2}\left(x+2\right)^{2}$$ |
|  |  |

* For $h=-2$, the $x$-intercepts are now located at $(-2, 0)$ and $(0,0)$
* The $x$-intercepts are located at $\left(h,0\right)$ and $\left(0,0\right)$
* Graphs where $m+n$ is even appear to be reflected in the $y$-axis, however this is not the case for graphs where $m+n$ is odd (Note: this could be extended into exploring odd and even functions)
* The number of turning points and points of inflection are the same as when $h$ is positive, however their locations are different
1. Create an equation of a graph of the form $y=x^{m}\left(h-x\right)^{n}$ that satisfies the following conditions:

Graph 1: Has an $x$-intercept and turning point at $x=3$

One possible answer: $y=x\left(3-x\right)^{2}$



Anything of the form $y=x^{m}\left(3-x\right)^{n}$ where $m\in N$ and $n\in 2N$ will work

Graph 2:

* Has at least 2 turning points
* Has a negative $x$-intercept

Possible answers could include:

|  |  |
| --- | --- |
| $$y=x^{2}\left(-1-x\right)^{2}$$ | $$y=x^{2}\left(-2-x\right)^{4}$$ |
| $$y=x^{2}(-2-x)$$ |  |

Graph 3:

* Has a stationary point of inflection
* Has a negative $x$-intercept

Possible answers could include:

|  |  |
| --- | --- |
| $$y=x^{3}(-2-x)$$ | $$y=x\left(-3-x\right)^{3}$$ |

**Component 3**

In this component you will consider graphs of the form $y=(x^{m}-a)(x^{n}-b)$,
where $a,b$ are non-zero real numbers and $m,n\in \left\{1, 2, 3\right\}$.

1. Consider the family of curves of the form $y=\left(x^{m}-1\right)\left(x^{n}-8\right)$ where $2\leq m+n\leq 4$.
Investigate how $m$ and $n$ affect:
* The location and number of axial intercepts
* The behaviour of the graph
* (The number of turning points/stationary points of inflection)

|  |  |  |  |
| --- | --- | --- | --- |
| $$m=1, n=1$$ | $$y=(x-1)(x-8)$$ | $$m=1, n=2$$ | $$y=(x-1)(x+1)(x-8)$$ |
|  |  |
| $$m=2, n=1$$ | $$y=(x-1)(x-\sqrt{8})(x+\sqrt{8})$$ | $$m=1, n=3$$ | $$y=(x-1)(x^{2}+x+1)(x-8)$$ |
|  |  |
| $$m=3, n=1$$ | $$y=(x-1)(x-2)(x^{2}+2x+4)$$ |  |  |
|  |  |

**The location and number of axial intercepts**

* In all cases, there is an $x$-intercept at $x=1$ (at the point $\left(1, 0\right)$)
* In all cases, there is a $y$-intercept at $y=8$ (at the point $\left(0, 8\right)$)
* In all cases, there two positive $x$-intercepts
* When $m+n=2$ or $4$ (or $m+n\in 2N$) there are only 2 $x$-intercepts, located at $\left(1,0\right)$ and $\left(8^{\frac{1}{m}},0\right)$ or $\left(\sqrt[m]{8},0\right)$
* When $m+n=3$ ($m+n\in 2N+1)$ there are three $x$-intercepts, since the factors are one that is linear and one that is a difference of perfect squares
* When $n>m$, the $x$-intercepts are at $(\pm 1, 0)$ and $\left(8^{\frac{1}{m}},0\right)$
* When $m>n$, the $x$-intercepts are at $\left(1,0\right)$ and $\left(\pm 8^{\frac{1}{m}}, 0\right)$

**The behaviour of the graph**

* When $m+n=2$ or $4$, the graph has a quadratic or quartic shape ($'U^{'}$ shape)
* When $m+n=3$, the graph has a cubic shape

**The number of turning points/stationary points of inflection**

* When $m+n=2$, there is one turning point
* When $m+n=3$, there are two turning points (and one non-stationary point of inflection)
* When $m+n=4$, there is one turning point and one point of inflection
1. Consider the family of curves of the form $y=\left(x^{m}-1\right)\left(x^{n}-a\right)$ where $2\leq m+n\leq 4$ and $a$ is a non-negative real number.
Investigate how $a, m$ and $n$ affect:
* The location and number of axial intercepts
* The behaviour of the graph
* (The number of turning points/stationary points of inflection)

**End of Investigation**